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TECHNICAL NOTE 3791

ON A METHOD FOR OPTIMIZATION OF TIME-VARYING LINEAR SYSTEMS WITH NONSTATIONARY INPUTS

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SUMMARY

By means of examples, a new method is illustrated for optimizing, over a finite interval of time, time-varying systems with stationary or nonstationary statistical inputs. The method depends on the correlation functions being of a certain type, which, fortunately, is the type found in a large number of practical problems of importance.

INTRODUCTION

Recently, particularly because of their application to missile and fire-control systems, considerable attention has been paid to optimization techniques for systems having statistical inputs. Historically, such techniques begin with Wiener's theory (ref. 1) of the optimum design of constant coefficient linear systems. To apply this theory it is required that the statistical properties of both the messages and the noise do not vary with time; quantities having this invariance property are called "stationary." This requirement, that messages and noise be stationary, is quite restrictive, eliminating from consideration even such a simple problem as the optimal determination of the position of a gun target, say, when the target is moving with constant speed in a straight line and the measurements are corrupted by noise. It seems desirable, therefore, to eliminate this hypothesis, if possible.

In reference 2, Booton attempted to generalize the Wiener theory to include nonstationary inputs and time-varying linear operations. Booton arrived at an integral equation, the solution of which would be the impulse response of the optimum system. However, no method for solving the equation was given.

A new method for solving integral equations such as the one Booton derived was recently discovered (unpublished). This method requires that the correlation functions which arise be of a certain form; fortunately, this requirement is very frequently satisfied in practice.

The purpose of this report is the illustration by means of examples of the application of this method. Since nonstationary optimization problems may lead to a time-varying linear system as the optimum, the report begins with a brief discussion of the perhaps unfamiliar idea of a time-varying transfer function. The report proceeds with an outline of a variant of the theory of reference 2, itemized, step-by-step procedures for solving the integral equation for the impulse response of the optimum system, and some relatively simple examples. Techniques for determining the differential equations of the optimum system from the impulse response are also shown in the examples. It is hoped that a study of these examples will be sufficient introduction to the subject to allow the method to be applied to more difficult practical problems.

PRELIMINARY DISCUSSION

TIME-VARYING TRANSFER FUNCTIONS

The meaning of the impulse response $g(t - \tau)$ of a time-invariant transfer function is well known. If the transfer function in question is of the form

$$\frac{f(p)}{F(p)}$$

where p denotes the operator d/dt, and f and F are polynomials, then $g(t - \tau)$ is the response of this system at time t to an impulse applied when $t = \tau$; that is,

$$F(p)g(t - \tau) = f(p)\delta(t - \tau)$$

where $\delta(t - \tau)$ symbolizes the Dirac δ function (ref. 3). The most important property of this impulse response is that if $i(t)$ is any input to the system, then the output in response to $i(t)$ is

$$\int_{-\infty}^{t} g(t - \tau)i(\tau)d\tau \tag{1}$$

For time-varying systems, the idea of an impulse response also exists. Consider the transfer function

$$\frac{f(t,p)}{F(t,p)}$$

The impulse response $g(t,\tau)$ for this system then satisfies the equation

$$F(t,p)g(t,\tau) = f(t,p)\delta(t - \tau)$$

It should be noted that for time-varying systems, the impulse response is some function of t and τ, and does not depend on their difference alone. The fundamental property (1) carries over to time-varying systems. If $i(t)$ is any input to the system, the output is

$$\int_{-\infty}^{t} g(t,\tau)i(\tau)d\tau \tag{2a}$$

As an example of this, consider the simple system with transfer function

$$\frac{1}{p + t^2}$$

The impulse response satisfies the equation

$$\frac{\partial}{\partial t} g(t,\tau) + t^2 g(t,\tau) = \delta(t - \tau)$$

It can be shown that this means

$$g(t,\tau) = e^{\frac{\tau^3 - t^3}{3}} u(t - \tau)$$

(where $u(t)$ is the unit step function: $u(t) = 0$, $t < 0$, $u(t) = 1$, $t \geq 0$) so that the response to any input $i(t)$ is

$$e^{-\frac{t^3}{3}} \int_{-\infty}^{t} e^{\frac{\tau^3}{3}} i(\tau)d\tau$$

Note, incidentally, that

$$g(t,\tau) = 0 \qquad \text{for} \quad t < \tau$$

This must always be the case for, if it were not, the system would be required to respond to an input before the latter occurred.

NOTATION AND DEFINITIONS

Although the ideas discussed herein have application to different fields, for ease of expression, the language of communication theorist will be used throughout - thus, we shall speak of "filters" designed to derive from an "input" an approximation to a "message" which has been corrupted by "noise."

Now, there will always be an ensemble of messages it is desired to transmit. Denote a typical message by the symbol $m(t; \alpha_1, \ldots, \alpha_M)$; the parameters α indicate which particular message of the ensemble is being considered. Denote the possible noise functions by $n(t; \beta_1, \ldots, \beta_N)$. The input to the filter we are attempting to design is defined by

$$i(t; \alpha_1, \ldots, \alpha_M, \beta_1, \ldots, \beta_N) = m(t; \alpha_1, \ldots, \alpha_M) + n(t; \beta_1, \ldots, \beta_N)$$

If $f(t; \alpha_1, \ldots, \alpha_M, \beta_1, \ldots, \beta_N)$ is any function of t and the parameters α and β, we shall mean by

$$\mathrm{Av}\left\{f(t; \alpha_1, \ldots, \alpha_M, \beta_1, \ldots, \beta_N)\right\}$$

the average of f with respect to the α's and β's, that is, with respect to the ensemble.

For later use, we now define the following correlation functions:

$$\left.\begin{aligned}
\varphi_{mm}(t,\tau) &= \mathrm{Av}\left\{m(t; \alpha_1, \ldots, \alpha_M)m(\tau; \alpha_1, \ldots, \alpha_M)\right\} \\
\varphi_{mi}(t,\tau) &= \mathrm{Av}\left\{m(t; \alpha_1, \ldots, \alpha_M)i(\tau; \alpha_1, \ldots, \alpha_M, \beta_1, \ldots, \beta_N)\right\} \\
\varphi_{ii}(t,\tau) &= \mathrm{Av}\left\{i(t; \alpha_1, \ldots, \alpha_M, \beta_1, \ldots, \beta_N)i(\tau; \alpha_1, \ldots, \alpha_M, \beta_1, \ldots, \beta_N)\right\}
\end{aligned}\right\} \quad (3)$$

THE INTEGRAL EQUATION FOR THE OPTIMUM

In reference 2, Booton, using the methods of reference 4, derived an integral equation for an optimum filter. Although the methods to be discussed are applicable to Booton's equation also, we shall apply it to a slightly different one. The reason for this is that Booton considered the general case when the inputs to the filter have no beginning in time, so that the system has been in operation infinitely long. Of course, in all real situations, a starting point exists - a time when the telephone is first picked up or the missile is first fired, etc. In view of this, the simplifying assumption that the inputs are nonexistent (zero) up to a certain time will be made here. By an appropriate choice of the time scale, this distinguished instant may be made zero. In this case, the response (2) of a system with impulse response $g(t,\tau)$ to the input $i(t)$ reduces to

$$\int_0^t g(t,\tau)i(\tau)d\tau \tag{2b}$$

Now, it is desired to filter the inputs to give as good as possible a representation of the messages; that is, it is desired to specify an impulse response $g(t,\tau)$ such that, in accordance with (2b)

$$\int_0^t g(t,\tau)i(\tau;\ \alpha_1,\ .\ .\ .,\ \alpha_M,\ \beta_1,\ .\ .\ .,\ \beta_N)d\tau$$

is as close as possible to $m(t;\ \alpha_1,\ .\ .\ .,\ \alpha_M)$. The expressions "as good as possible" and "as close as possible" remain to be defined. They will be taken to mean that the mean square error is as small as possible - that is, $g(t,\tau)$ is to be chosen such that

$$E^2(t) \equiv \mathrm{Av}\left\{\left[m(t;\ \alpha_1,\ .\ .\ .,\ \alpha_M) - \int_0^t g(t,\tau)i(\tau;\ \alpha_1,\ .\ .\ .,\ \alpha_M,\ \beta_1,\ .\ .\ .,\ \beta_N)d\tau\right]^2\right\} \tag{4a}$$

is a minimum. It should be noted, incidentally, that the mean square error E^2 is a function of t since the average is taken with respect to the ensemble with t fixed.

Now, it is not hard to show that equations (3) and (4a) together give

$$E^2(t) = \varphi_{mm}(t,t) - 2\int_0^t g(t,\tau)\varphi_{mi}(t,\tau)\,d\tau + \int_0^t g(t,\tau)\int_0^t g(t,\sigma)\varphi_{ii}(\tau,\sigma)\,d\sigma\,d\tau \tag{5a}$$

and, by a method exactly parallel to the one used in references 2 and 4, it can be shown that this error is a minimum if and only if $g(t,\tau)$ satisfies the linear integral equation

$$\varphi_{mi}(t,\tau) = \int_0^t g(t,\sigma)\varphi_{ii}(\tau,\sigma)\,d\sigma\ , \quad \text{for} \quad 0 \leq \tau \leq t \tag{6a}$$

Inserting equation (6a) into (5a) gives for the minimum mean square error:

$$E^2{}_{min}(t) = \varphi_{mm}(t,t) - \int_0^t g(t,\tau)\varphi_{mi}(t,\tau)\,d\tau \tag{7a}$$

Not infrequently, it is desired to generalize this technique in the sense that it is desired to obtain from the input some quantity dependent on the message but not the message itself. Thus, for example, it might be desired to predict the message h seconds hence or to integrate or differentiate the message. To do this, we minimize

$$E^2(t) = \mathrm{Av}\left\{\left[\mu(t;\ \alpha_1,\ .\ .\ .,\ \alpha_M) - \int_0^t g(t,\tau)i(\tau;\ \alpha_1,\ .\ .\ .,\ \alpha_M,\ \beta_1,\ .\ .\ .,\ \beta_N)\,d\tau\right]^2\right\} \tag{4b}$$

where μ is the quantity we wish to obtain from the input. Thus, if it is desired to integrate the message, $\mu(t;\ \alpha_1,\ .\ .\ .,\ \alpha_M)$ is the integral of $m(t;\ \alpha_1,\ .\ .\ .,\ \alpha_M)$; if it is desired to differentiate the message, μ is the time derivative of m, etc. Defining

$$\varphi_{\mu\mu}(t,\tau) = \mathrm{Av}\left\{\mu(t;\ \alpha_1,\ .\ .\ .,\ \alpha_M)\mu(\tau;\ \alpha_1,\ .\ .\ .,\ \alpha_M)\right\}$$

$$\varphi_{\mu i}(t,\tau) = \mathrm{Av}\left\{\mu(t;\ \alpha_1,\ .\ .\ .,\ \alpha_M)i(\tau;\ \alpha_1,\ .\ .\ .,\ \alpha_M,\ \beta_1,\ .\ .\ .,\ \beta_N)\right\}$$

we find that the generalization of equation (5a) is

$$E^2(t) = \varphi_{\mu\mu}(t,t) - 2\int_0^t g(t,\tau)\varphi_{\mu i}(t,\tau)d\tau + \int_0^t g(t,\tau)\int_0^t g(t,\sigma)\varphi_{ii}(\tau,\sigma)d\sigma\, d\tau \tag{5b}$$

The equation from which the optimum impulse response is to be determined is

$$\varphi_{\mu i}(t,\tau) = \int_0^t g(t,\sigma)\varphi_{ii}(\tau,\sigma)d\sigma \ , \qquad \text{for} \quad 0 \leq \tau \leq t \tag{6b}$$

and the minimum mean square error is

$$E^2{}_{min}(t) = \varphi_{\mu\mu}(t,t) - \int_0^t g(t,\tau)\varphi_{\mu i}(t,\tau)d\tau \tag{7b}$$

Since equation (6b) is the more general, in the sense that it reduces to (6a) when μ is set equal to m, all considerations which follow will refer to this equation. If the problem is simply one of filtering, it will only be necessary to set $\mu = m$ in all that follows.

ASSUMPTIONS

The basic assumptions which will be made in this paper are the following:

(i) The functions $\varphi_{ii}(t,\tau)$ and $\varphi_{\mu i}(t,\tau)$ are of the form[1]

$$\left.\begin{aligned} \varphi_{ii}(t,\tau) &= \begin{cases} \sum_1^P a_p(t)b_p(\tau) \ , & \tau \leq t \\ \sum_1^P a_p(\tau)b_p(t) \ , & \tau > t \end{cases} \\ \varphi_{\mu i}(t,\tau) &= \sum_1^P c_p(t)b_p(\tau) \ , \qquad \tau \leq t \end{aligned}\right\} \tag{8a}$$

[1]More generally, φ_{ii} is allowed to involve δ functions depending on $t - \tau$ (white noise); however, as will be seen, this is a limiting case of the form (8).

(ii) If $a_p(t)$ and $b_p(t)$ have the same meaning as in (i), then the quantity

$$w = \sum_{1}^{P} [a_p(t)b_p(\tau) - a_p(\tau)b_p(t)]$$

is a function of $t - \tau$ alone:

$$w = w(t - \tau)$$

The following remarks on these assumptions are pertinent. First, in almost all cases of importance, φ_{ii} and $\varphi_{\mu i}$ either will be of the form (8a) or may be successfully approximated by functions of this form; this can be seen from the definition (3) of φ_{ii} and $\varphi_{\mu i}$ as averages of products of functions of t and functions of τ. Second, note that the expression for $\varphi_{ii}(t,\tau)$ when $\tau > t$ is a necessary consequence of the assumed form when $\tau \leq t$ and is not a separate assumption since, by its definition (3), $\varphi_{ii}(t,\tau) = \varphi_{ii}(\tau,t)$. Third, under certain circumstances where the assumption (ii) does not apply, alternative methods still exist.

EXAMPLES

Five examples will be given. They were chosen primarily to illustrate different aspects of the problem of solving the fundamental integral equation. In order to do this most successfully, the examples have been broken up into three groups: examples involving white noise, examples with other types of noise, and a third class of examples, the necessity for which will appear as we proceed.

WHITE NOISE

White noise is said to occur when the autocorrelation function of the noise has the form

$$\varphi_{nn}(t,\tau) = N\delta(t - \tau)$$

where $\delta(t - \tau)$ denotes the Dirac δ function. As will be seen in the next section, this kind of noise can be considered as a limiting case of "continuous" noise, and so a problem involving white noise can always be solved by setting up and solving another problem involving continuous noise and then taking limits. A simpler procedure, however, would be to describe a method for solving problems with white noise directly.

In order to do so, some assumptions resembling (i) and (ii) above must be made. We shall assume the following:

(i') The functions of $\varphi_{ii}(t,\tau)$ and $\varphi_{\mu i}(t,\tau)$ have the form

$$\left.\begin{aligned} \varphi_{ii}(t,\tau) &= \begin{cases} \sum_{1}^{Q} a_q(t)b_q(\tau) + N\delta(t-\tau), & \tau \leq t \\ \sum_{1}^{Q} a_q(\tau)b_q(t) + N\delta(\tau-t), & \tau > t \end{cases} \\ \varphi_{\mu i}(t,\tau) &= \sum_{1}^{Q} c_q(t)b_q(\tau), \qquad\qquad\qquad \tau \leq t \end{aligned}\right\} \tag{8b}$$

(ii') If $a_q(t)$ and $b_q(t)$ have the same meaning as in (i'), then the quantity

$$v = \sum_{1}^{Q} \left[a_q(t)b_q(\tau) - a_q(\tau)b_q(t)\right]$$

is a function of $t - \tau$ alone:

$$v = v(t - \tau)$$

With these assumptions, a method for solving the integral equation when the noise is white can be outlined.

STEP 1. Let $B_q(s)$ and $V(s)$ denote the Laplace transforms of $b_q(t)$ and $v(t)$, respectively. Then, find the quantities

$$\Gamma_q(s) = \frac{B_q(s)}{N + V(s)}, \qquad n = 1, \ldots, Q$$

and let $\gamma_q(t)$ denote the inverse Laplace transform of $\Gamma_q(s)$.

STEP 2. Compute

$$I_{pq}(t) = \int_{0}^{t} a_p(\tau)\gamma_q(\tau)d\tau$$

and solve the equations

$$\left.\begin{array}{l}
[1 + I_{11}(t)]g_1(t) + \quad I_{12}(t)\, g_2(t) + \ldots + \quad I_{1Q}(t)\, g_Q(t) = c_1(t) \\
I_{21}(t)\, g_1(t) + [1 + I_{22}(t)]g_2(t) + \ldots + \quad I_{2Q}(t)\, g_Q(t) = c_2(t) \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
I_{Q1}(t)\, g_1(t) + \quad I_{Q2}(t)\, g_2(t) + \ldots + [1 + I_{QQ}(t)]g_Q(t) = c_Q(t)
\end{array}\right\} \tag{9}$$

for the functions $g_q(t)$, $q = 1, \ldots, Q$.

Finally, let $u(t)$ denote the unit step function:

$$u(t) = \begin{cases} 0\,, & t < 0 \\ 1\,, & t \geq 0 \end{cases}$$

Then, the function

$$g(t,\tau) = u(t - \tau) \sum_{1}^{Q} g_q(t)\gamma_q(\tau) \tag{10}$$

satisfies the integral equation for the optimum impulse response.

We now apply this to two examples.

Example I

To begin, consider the following question, equivalent to the one already posed in the Introduction. A collection of particles leaves the origin at a certain time; thereafter, each particle moves with constant (unknown) velocity. Assuming our measurements to be corrupted by white noise, what is the best way to determine the positions of the particles at any succeeding time?

Calculation of the correlation functions.- Let α denote the speed of a typical particle, and let $\overline{\alpha^2}$ denote the mean square speed, averaged over all the particles. Although we do not assume α to be known for any particular particle, we do assume the mean square speed $\overline{\alpha^2}$ to be known.

We choose the time that the particles leave the origin as zero, so that the messages (i.e., the particle positions) are given by

$$m(t;\alpha) = \alpha t\,, \qquad t \geq 0$$

Hence, we can compute

$$\varphi_{mm}(t,\tau) = \mathrm{Av}\left\{(\alpha t)(\alpha\tau)\right\}$$

$$= \overline{\alpha^2}\, t\tau$$

It is assumed here that the noise is white. It follows that it has zero mean. If it is assumed (quite reasonably) that the noise and the message are independent, it follows that $\varphi_{mn} = \varphi_{nm} = 0$, and so

$$\begin{aligned}
\varphi_{mi}(t,\tau) &= \mathrm{Av}\left\{m(t)i(\tau)\right\} \\
&= \mathrm{Av}\left\{m(t)m(\tau)\right\} + \mathrm{Av}\left\{m(t)n(\tau)\right\} \\
&= \varphi_{mm}(t,\tau) + \varphi_{mn}(t,\tau) \\
&= \overline{\alpha^2}\, t\tau
\end{aligned}$$

Similarly,

$$\begin{aligned}
\varphi_{ii}(t,\tau) &= \varphi_{mm}(t,\tau) + \varphi_{nn}(t,\tau) \\
&= \overline{\alpha^2}\, t\tau + N\delta(t - \tau)
\end{aligned}$$

The functions φ_{ii} and $\varphi_{\mu i}(= \varphi_{mi})$ clearly have the form (8b) with $Q = 1$. In fact, we may set

$$a_1(t) = \overline{\alpha^2}\, t, \quad b_1(t) = t\ , \quad c_1(t) = \overline{\alpha^2}\, t \tag{11}$$

<u>The integral equation and its solution</u>.- To apply the method outlined on pages 11 and 12, it is unnecessary actually to write down the integral equation (6). However, it is convenient to have this equation, since it can be used to check the answer. With the correlation functions given in the preceding paragraph, the integral equation becomes

$$\overline{\alpha^2}\, t\tau = \overline{\alpha^2}\,\tau\int_0^t \sigma g(t,\sigma)\,d\sigma + Ng(t,\tau)\ , \qquad \text{for } 0 \leq \tau \leq t \tag{12}$$

The last term in this equation arises from the δ function occurring in the expression for φ_{ii}; it is not an integral since the fundamental property of the δ function

$$\int_0^t g(t,\sigma)\delta(\tau-\sigma)d\sigma = g(t,\tau) \qquad \text{if} \quad 0 \le \tau \le t$$

has been used.

We now apply the method outlined earlier to solve equation (12). First, we check to see if assumption (ii') is fulfilled. It is, since

$$\begin{aligned} v &= a_1(t)b_1(\tau) - a_1(\tau)b_1(t) \\ &= \overline{\alpha^2}t \cdot \tau - \overline{\alpha^2}\tau \cdot t \\ &= 0 \end{aligned}$$

which may be considered as a function of $t - \tau$, notably that function which is always zero.

We now proceed with the solution.

STEP 1. Since $b_1(t)$ is as given by (11), we have

$$B_1(s) = \frac{1}{s^2}$$

Also, since $v(t) = 0$, $V(s) = 0$. Thus, we define

$$\Gamma_1(s) = \frac{1}{Ns^2}$$

and inverting

$$\gamma_1(t) = \frac{t}{N}$$

STEP 2. Since in this example, $Q = 1$, we need only find

$$I_{11}(t) = \int_0^t a_1(\tau)\gamma_1(\tau)d\tau$$

$$= \int_0^t \overline{\alpha^2}\tau \frac{\tau}{N}\, d\tau$$

$$= \frac{\overline{\alpha^2}}{3N}\, t^3$$

Hence, equations (9) become

$$\left(1 + \frac{\overline{\alpha^2}}{3N}\, t^3\right)g_1(t) = \overline{\alpha^2}t$$

that is,

$$g_1(t) = \frac{\overline{\alpha^2}t}{1 + (\overline{\alpha^2}/3N)t^3}$$

Therefore, by equation (10), the desired solution of the integral equation (12) can be written:

$$g(t,\tau) = \frac{\overline{\alpha^2}t\tau}{N + (\overline{\alpha^2}/3)t^3}\, u(t - \tau) \tag{13}$$

It should be noted that $g(t,\tau)$ vanishes if the impulse to which it is the response occurs at the time $\tau = 0$. This is so because we have assumed that the initial position of the particles is known precisely - that is, that at zero time the particles are all at the origin. This is one of the many respects in which the present simplified example does not describe the true situation for an actual gun platform.

The result can be checked by substitution into equation (12). In fact, for $0 \leq \tau \leq t$,

$$\overline{\alpha^2}\tau \int_0^t \sigma g(t,\sigma)d\sigma + Ng(t,\tau) = \overline{\alpha^2}\tau \int_0^t \frac{\overline{\alpha^2}t}{N + (\overline{\alpha^2}/3)t^3}\sigma^2 d\sigma + \frac{N\overline{\alpha^2}t\tau}{N + (\overline{\alpha^2}/3)t^3}$$

$$= \overline{\alpha^2}\tau \frac{\overline{\alpha^2}t}{N + (\overline{\alpha^2}/3)t^3}\frac{t^3}{3} + \frac{N\overline{\alpha^2}t\tau}{N + (\overline{\alpha^2}/3)t^3}$$

$$= \overline{\alpha^2}t\tau$$

which is as it should be.,

The error.- The minimum error can be found by using equation (7a); in fact,

$$E^2_{min}(t) = \overline{\alpha^2}t^2 - \int_0^t \frac{\overline{\alpha^2}t\tau}{N + (\overline{\alpha^2}/3)t^3}\overline{\alpha^2}t\tau\, d\tau$$

$$= \overline{\alpha^2}t^2 - \frac{(\overline{\alpha^2})^2 t^2}{N + (\overline{\alpha^2}/3)t^3}\frac{t^3}{3}$$

$$= \frac{N\overline{\alpha^2}t^2}{N + (\overline{\alpha^2}/3)t^3}$$

Note that the error approaches zero as $t \rightarrow \infty$.

The system differential equation.- We shall now derive a differential equation which relates the input and output of the optimum system. Thus, we seek two functions, $f(t,p)$ and $F(t,p)$, such that

$$F(t,p)g(t,\tau) = f(t,p)\delta(t - \tau) \tag{14}$$

where $p = d/dt$. These functions are required to be polynomials in p. Also, for the impulse response (10), the order of the differential equation is always equal to the number of terms in the sum on the right-hand side of equation (10) - that is, Q. Thus, we write

$$F(t,p) = p^Q + \xi_{Q-1}(t)p^{Q-1} + \ldots + \xi_0(t) \tag{15}$$

Since there are no δ functions in the expression (10) for g, the order of f is at most $Q - 1$; hence,

$$f(t,p) = \eta_{Q-1}(t)p^{Q-1} + \ldots + \eta_0(t) \tag{16}$$

Our problem now is the determination of the functions ξ_k and η_k.

In example I, $Q = 1$, and so

$$F(t,p) = p + \xi_0(t)$$

$$f(t,p) = \eta_0(t)$$

Thus, the differential equation (14) becomes

$$\frac{\partial}{\partial t} g(t,\tau) + \xi_0(t)g(t,\tau) = \eta_0(t)\delta(t-\tau) \tag{17}$$

Now, the function $\xi_0(t)$ can be found immediately, since $\delta(t-\tau) = 0$ for $t > \tau$. Thus, for $t > \tau$ we have

$$\xi_0(t) = -\frac{(\partial/\partial t)g(t,\tau)}{g(t,\tau)}$$

Hence, from equation (13),

$$\xi_0(t) = \frac{2\overline{\alpha^2}t^3 - 3N}{t(\overline{\alpha^2}t^3 + 3N)}$$

and the differential equation (17) becomes

$$\frac{\partial}{\partial t} g(t,\tau) + \frac{2\overline{\alpha^2}t^3 - 3N}{t(\overline{\alpha^2}t^3 + 3N)} g(t,\tau) = \eta_0(t)\delta(t-\tau) \tag{18}$$

To compute $\eta_0(t)$, substitute $g(t,\tau)$ as given by equation (13) into (18). This gives

$$\frac{\overline{\alpha^2}t\tau}{N + (\overline{\alpha^2}/3)t^3}\,\delta(t - \tau) = \eta_o(t)\delta(t - \tau)$$

using the fact that $(d/dt)u(t - \tau) = \delta(t - \tau)$. Now,

$$\frac{\overline{\alpha^2}t\tau}{N + (\overline{\alpha^2}/3)t^3}\,\delta(t - \tau) = \frac{\overline{\alpha^2}t^2}{N + (\overline{\alpha^2}/3)t^3}\,\delta(t - \tau)$$

since for $\tau \neq t$, both sides of this equation are zero, while for $\tau = t$, they are obviously equal. Hence

$$\eta_o(t)\delta(t - \tau) = \frac{\overline{\alpha^2}t^2}{N + (\overline{\alpha^2}/3)t^3}\,\delta(t - \tau)$$

that is,

$$\eta_o(t) = \frac{\overline{\alpha^2}t^2}{N + (\overline{\alpha^2}/3)t^3}$$

Therefore, finally, we may say if $i(t)$ is any input to the optimum system and $x(t)$ the corresponding output, $x(t)$ and $i(t)$ are related by the equation

$$(\overline{\alpha^2}t^3 + 3N)\dot{x} + \frac{2\overline{\alpha^2}t^3 - 3N}{t}\,x = 3\overline{\alpha^2}t^2 i$$

Example II

For our second example, we consider an air-to-air missile attack situation. Since the purpose of this report is not the development of an optimum missile guidance system, the problem will be simplified to the point where the features of the method are not obscured by the special requirements of the problem. Although the resulting situation is unrealistic, it does still possess some characteristics of interest.

A two-dimensional situation, wherein the missile and the target move in one plane, will be assumed. The missile is fired at a certain time, called zero, at which it is assumed the target position is known precisely.

A coordinate system can then be fixed in space in such a way that the target displacement initially is zero. It is assumed that after a certain time T the missile is moving so slowly it can no longer do any damage; thus, if the target is to take effective evasive action, it must do so in the first T seconds after the missile is fired. Finally, it is assumed that the evasive maneuver concerned is a step of magnitude α_1, the jump occurring at time α_2. Thus, we are considering the following ensemble of messages (target displacement):

$$m(t;\alpha_1,\alpha_2) = \begin{cases} 0\ , & t < \alpha_2 \\ \alpha_1\ , & t > \alpha_2 \end{cases}$$

Calculation of the correlation functions.- Let us assume a certain probability distribution of magnitudes α_1 is known; thus, we assume knowledge of a function $f(x)$ such that the probability that α_1 lies between x and $x + dx$ is $f(x)dx$. As for the time α_2 at which the jump occurs, let it be assumed that this quantity has equal likelihood of taking on any value in the interval $0 \leq \alpha_2 \leq T$. Then, for $0 \leq \tau \leq t$,

$$\varphi_{mm}(t,\tau) = \mathrm{Av}\left\{m(t;\alpha_1,\alpha_2)m(\tau;\alpha_1,\alpha_2)\right\}$$

$$= \frac{1}{T}\int_{-\infty}^{+\infty} f(\alpha_1) \int_0^T m(t;\alpha_1,\alpha_2)m(\tau;\alpha_1,\alpha_2)d\alpha_2 d\alpha_1$$

Since $m(\tau;\alpha_1,\alpha_2) = 0$ for $\alpha_2 > \tau$, this simplifies to

$$\varphi_{mm}(t,\tau) = \frac{1}{T}\int_{-\infty}^{+\infty} f(\alpha_1) \int_0^\tau \alpha_1^2 d\alpha_2 d\alpha_1$$

$$= \frac{\tau}{T}\int_{-\infty}^{+\infty} f(\alpha_1)\alpha_1^2 d\alpha_1$$

The integral occurring here is just the mean square value of α_1, whatever the distribution function may be; consequently, denoting the mean square value of α_1 by $\overline{\alpha_1^2}$, we obtain

$$\varphi_{mm}(t,\tau) = \frac{\overline{\alpha_1^2}}{T}\tau \qquad \text{for} \quad \tau \leq t$$

Now, it is obvious from the definition (3) that the autocorrelation function φ_{mm} is symmetric in t and τ: $\varphi_{mm}(t,\tau) = \varphi_{mm}(\tau,t)$. Hence,

$$\varphi_{mm}(t,\tau) = \frac{\overline{\alpha_1^2}}{T} t \qquad \text{for} \quad \tau > t$$

Once again, we assume the noise white and independent of message, so that

$$\varphi_{mi}(t,\tau) = \varphi_{mm}(t,\tau) = \begin{cases} \dfrac{\overline{\alpha_1^2}}{T}\tau & \text{for} \quad \tau \leq t \\ \dfrac{\overline{\alpha_1^2}}{T} t & \text{for} \quad \tau > t \end{cases}$$

$$\varphi_{ii}(t,\tau) = \varphi_{mm}(t,\tau) + \varphi_{nn}(t,\tau) = \begin{cases} \dfrac{\overline{\alpha_1^2}}{T}\tau + N\delta(t-\tau) & \text{for} \quad \tau \leq t \\ \dfrac{\overline{\alpha_1^2}}{T} t + N\delta(\tau - t) & \text{for} \quad \tau > t \end{cases}$$

Comparing this form with equation (8b), we see that assumption (i') will be fulfilled if we choose $Q = 1$ and

$$a_1(t) = \frac{\overline{\alpha_1^2}}{T}, \qquad b_1(t) = t, \qquad c_1(t) = \frac{\overline{\alpha_1^2}}{T}$$

<u>The integral equation and its solution</u>.- Equation (6a) becomes, in this case,

$$\frac{\overline{\alpha_1^2}}{T}\tau = \frac{\overline{\alpha_1^2}}{T}\int_0^\tau \sigma g(t,\sigma)d\sigma + \frac{\overline{\alpha_1^2}}{T}\tau\int_\tau^t g(t,\sigma)d\sigma + Ng(t,\tau), \qquad 0 \leq \tau \leq t$$

Before proceeding to the solution, we check that assumption (ii') is fulfilled. We have

$$v = a_1(t)b_1(\tau) - a_1(\tau)b_1(t)$$

$$= \frac{\overline{\alpha_1^2}}{T}\tau - \frac{\overline{\alpha_1^2}}{T}t$$

$$= -\frac{\overline{\alpha_1^2}}{T}(t - \tau)$$

Hence it is fulfilled, with $v(t) = -\frac{\overline{\alpha_1^2}}{T}t$.

We now apply the method outlined earlier.

STEP 1. We find first that

$$B_1(s) = \frac{1}{s^2}$$

$$V(s) = -\frac{\overline{\alpha_1^2}}{T}\frac{1}{s^2}$$

Hence, we define

$$\Gamma_1(s) = \frac{\frac{1}{s^2}}{N - \frac{\overline{\alpha_1^2}}{Ts^2}}$$

$$= \frac{1}{Ns^2 - \frac{\overline{\alpha_1^2}}{T}}$$

Consequently,

$$\gamma_1(t) = \frac{\sinh \lambda t}{N\lambda}$$

where

$$\lambda^2 = \frac{\overline{\alpha_1^2}}{NT}$$

STEP 2. We have

$$I_{11}(t) = \int_0^t a_1(\tau)\gamma_1(\tau)d\tau$$

$$= \int_0^t \frac{\overline{\alpha_1^2}}{T} \frac{\sinh \lambda\tau}{N\lambda} d\tau$$

$$= \cosh \lambda t - 1$$

Hence, equations (9) become

$$g_1(t)\cosh \lambda t = \frac{\overline{\alpha_1^2}}{T} = N\lambda^2$$

that is,

$$g_1(t) = N\lambda^2 \text{sech } \lambda t$$

Therefore, from equation (10), the optimum impulse response can be written:

$$g(t,\tau) = \lambda u(t - \tau)\text{sech } \lambda t \sinh \lambda\tau$$

The error.- The minimum mean square error may be found from equation (7a) to be

$$E^2{}_{min}(t) = \frac{\overline{\alpha_1^2}}{T} t - \int_0^t \lambda \text{ sech } \lambda t \sinh \lambda\tau \frac{\overline{\alpha_1^2}}{T} \tau \, d\tau$$

$$= N\lambda \tanh \lambda t$$

The system differential equation.- By exactly the same method as was used in example I, the differential equation relating the output of the optimum system to the input can be shown to be

$$\dot{x} + (\lambda \tanh \lambda t)x = (\lambda \tanh \lambda t)i$$

OTHER TYPES OF NOISE

In this section, we shall assume the autocorrelation of the inputs not to contain δ functions. With this assumption and assumptions (i) and (ii) of pages 7 and 8, the method for solving the integral equation can be outlined as follows:

STEP 1. Let $B_p(s)$ and $W(s)$ denote the Laplace transforms of $b_p(t)$ and $w(t)$, respectively, and define

$$\Gamma_p(s) = \frac{B_p(s)}{W(s)}$$

Let $\gamma_p(t)$ denote the inverse Laplace transform of $\Gamma_p(s)$.

STEP 2. Set

$$I_{pq}(t) = \int_0^t a_p(\tau)\gamma_q(\tau)d\tau$$

Let $d_p(t)$, $p = 1, \ldots, P$ be any P functions having the form

$$d_p(t) = c_p(t) - \sum_k h_k(t) \frac{d^k}{dt^k} a_p(t) \tag{19}$$

Solve the equations

$$\left.\begin{array}{l} [1 + I_{11}(t)]g_1(t) + I_{12}(t)\, g_2(t) + \ldots + I_{1P}(t)\, g_P(t) = d_1(t) \\ I_{21}(t)\, g_1(t) + [1 + I_{22}(t)]g_2(t) + \ldots + I_{2P}(t)\, g_P(t) = d_2(t) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ I_{P1}(t)\, g_1(t) + I_{P2}(t)\, g_2(t) + \ldots + [1 + I_{PP}(t)]g_P(t) = d_P(t) \end{array}\right\} \tag{20}$$

for the $g_q(t)$ as functions of the $d_p(t)$. Then, the function

$$g(t,\tau) = \sum_k h_k(t)\delta^{(k)}(t - \tau) + u(t - \tau)\sum_1^P g_\nu(t)\gamma_\nu(\tau) \tag{21}$$

will satisfy the integral equation (6), whatever the functions h_k may be.

STEP 3. Choose the functions h_k in such a way that $g(t,\tau)$ is as simple as possible, involving the least possible number of differentiations.

It is realized that the description of step 3 is rather vague, but examination of the examples which follow should clarify this.

We now turn to the examples.

Example III

This third example will be the same as the first, the only difference being in the noise.

The correlation functions.- As in example I, the autocorrelation of the messages has the form

$$\varphi_{mm}(t,\tau) = \overline{\alpha^2}t\tau \tag{22}$$

Let us assume that the noise is independent of the message and has zero mean. Then, as before

$$\begin{aligned}\varphi_{mi}(t,\tau) &= \varphi_{mm}(t,\tau) \\ &= \overline{\alpha^2}t\tau\end{aligned} \tag{23}$$

Finally, we shall take the autocorrelation of the noise to be

$$\varphi_{nn}(t,\tau) = Be^{-\beta|t-\tau|} \tag{24}$$

This expression approximates the autocorrelation of actual radar noise. The assumptions made imply that

$$\varphi_{ii}(t,\tau) = \overline{\alpha^2}t\tau + Be^{-\beta|t-\tau|}$$

We mention here that the noise described by equation (24) approaches white noise as B and β approach infinity; this will be shown below.

<u>The integral equation and its solution</u>.- Once again, it is not necessary to write down the integral equation, but we shall do so since it can be used to afford a useful check on the results. With the correlation functions of the preceding section, the integral equation (6a) becomes

$$\overline{\alpha^2}t\tau = \overline{\alpha^2}\tau \int_0^t \sigma g(t,\sigma)d\sigma + Be^{-\beta\tau}\int_0^\tau e^{\beta\sigma}g(t,\sigma)d\sigma +$$

$$Be^{\beta\tau}\int_\tau^t e^{-\beta\sigma}g(t,\sigma)d\sigma \;, \qquad 0 \leq \tau \leq t \tag{25}$$

We now solve this equation by the method outlined. Comparison of the expressions (22), (23), and (24) for the correlation functions with the expressions (8a) gives the results $P = 2$, and

$$a_1(t) = \overline{\alpha^2}t \qquad b_1(t) = t \qquad c_1(t) = \overline{\alpha^2}t$$

$$a_2(t) = Be^{-\beta t} \qquad b_2(t) = e^{\beta t} \qquad c_2(t) = 0$$

Hence,

$$\begin{aligned} w &= \overline{\alpha^2}t \cdot \tau - \overline{\alpha^2}\tau \cdot t + Be^{-\beta t} \cdot e^{\beta\tau} - Be^{-\beta\tau} \cdot e^{\beta t} \\ &= -2B \sinh \beta(t - \tau) \\ &= w(t - \tau) \end{aligned}$$

Thus, assumption (ii) is satisfied with

$$w(t) = -2B \sinh \beta t$$

STEP 1. Taking Laplace transforms, we have

$$B_1(s) = \frac{1}{s^2} \;, \qquad B_2(s) = \frac{1}{s - \beta}$$

$$W(s) = -\frac{2B\beta}{s^2 - \beta^2}$$

Therefore, we define

$$\Gamma_1(s) = -\frac{s^2 - \beta^2}{2B\beta}\frac{1}{s^2}$$

$$= \frac{1}{2B\beta}\left(\frac{\beta^2}{s^2} - 1\right)$$

$$\Gamma_2(s) = -\frac{s^2 - \beta^2}{2B\beta}\frac{1}{s - \beta}$$

$$= -\frac{s + \beta}{2B\beta}$$

Inverting the Laplace transform gives

$$\gamma_1(t) = \frac{1}{2B\beta}\left[\beta^2 t - \delta(t)\right]$$

$$\gamma_2(t) = \frac{1}{2B\beta}\left[\dot{\delta}(t) - \beta\delta(t)\right] \tag{26}$$

STEP 2. We have

$$I_{11}(t) = \frac{\overline{\alpha^2}}{2B\beta}\int_0^t \tau\left[\beta^2\tau - \delta(\tau)\right]d\tau$$

$$= \frac{\beta\overline{\alpha^2}}{6B}t^3$$

$$I_{12}(t) = \frac{\overline{\alpha^2}}{2B\beta}\int_0^t \tau\left[\dot{\delta}(\tau) - \beta\delta(\tau)\right]d\tau$$

$$= -\frac{\overline{\alpha^2}}{2B\beta}$$

since, in general,

$$\int_0^t f(\tau)\delta^{(n)}(\tau)d\tau = (-1)^n f^{(n)}(0)$$

$$I_{21}(t) = \frac{1}{2\beta}\int_0^t e^{-\beta\tau}[\beta^2\tau - \delta(\tau)]d\tau$$

$$= \frac{1}{2\beta}\left\{[1 - e^{-\beta t}(1 + \beta t)] - 1\right\}$$

$$= -e^{-\beta t}\,\frac{1 + \beta t}{2\beta}$$

$$I_{22}(t) = \frac{1}{2\beta}\int_0^t e^{-\beta\tau}[\dot{\delta}(\tau) - \beta\delta(\tau)]d\tau$$

$$= \frac{1}{2\beta}(\beta - \beta)$$

$$= 0$$

Consequently, equations (20) for g_1 and g_2 become

$$\left.\begin{aligned} \left(1 + \frac{\beta\overline{\alpha^2}}{6B}t^3\right)g_1(t) - \frac{\overline{\alpha^2}}{2B\beta}g_2(t) &= d_1(t) \\ -e^{-\beta t}\frac{1 + \beta t}{2\beta}g_1(t) + g_2(t) &= d_2(t) \end{aligned}\right\}$$

These equations can be solved to give

$$g_1(t) = 6\beta\,\frac{2B\beta d_1(t) + \overline{\alpha^2}\,d_2(t)}{2\beta^2(6B + \beta\overline{\alpha^2}t^3) - 3\overline{\alpha^2}(1 + \beta t)e^{-\beta t}}$$

$$g_2(t) = 2\beta\,\frac{3B(1 + \beta t)e^{-\beta t}d_1(t) + \beta(6B + \beta\overline{\alpha^2}t^3)d_2(t)}{2\beta^2(6B + \beta\overline{\alpha^2}t^3) - 3\overline{\alpha^2}(1 + \beta t)e^{-\beta t}} \qquad (27)$$

STEP 3. We have now arrived at the task of solidifying the earlier vague description of this step. The idea behind step 3 is very simple and was conceived in order to reduce the number of differentiations of noisy inputs.

Note first that if all functions h_k were taken to be zero, the solution (21) would contain only the second sum in that expression and so would resemble very closely our earlier solution (10) for the case of white noise. However, although the solution (10) of any problem involving white noise will never contain a differentiator,[2] this is not the case for other types of noise. In the present example III, the existence of a differentiator manifests itself through the occurrence of a term $\dot{\delta}(t)$ in the expression (26) for $\gamma_2(t)$.

Now, the function $\gamma_2(\tau)$ is multiplied in equation (21) by $g_2(t)$. Consequently, if g_2 could be made to be zero, the differentiator in γ_2 would be eliminated. From equation (27), we see that setting g_2 equal to zero gives an equation in the d's. According to equation (19), however, the d's are functions of the h's. Therefore, if the functions h_k are free and may be assigned at will, there is the possibility that $g_2(t)$ may be made to vanish. In this example, only one function (g_2) must be eliminated; consequently, there appears to be need for only one function h, and so we set

$$h_k(t) = 0 , \quad k \geq 1$$

leaving only $h_o(t)$.

Now, setting $g_2(t)$ equal to zero gives

$$3B(1 + \beta t)e^{-\beta t}d_1(t) + \beta(6B + \beta\overline{\alpha^2}t^3)d_2(t) = 0 \tag{28}$$

By virtue of equations (19) and the given values of the functions $a_n(t)$, we have

$$d_1(t) = \overline{\alpha^2}t[1 - h_o(t)]$$

$$d_2(t) = -Be^{-\beta t}h_o(t)$$

Therefore, equation (28) is equivalent to

$$3\overline{\alpha^2}t(1 + \beta t)[1 - h_o(t)] - \beta(6B + \beta\overline{\alpha^2}t^3)h_o(t) = 0$$

[2] This follows from the fact that when the noise is white, the integral equation is of the second kind (cf. (12)).

that is,

$$h_o(t) = \frac{3\overline{\alpha^2}t(1 + \beta t)}{6B\beta + 3\overline{\alpha^2}t + 3\overline{\alpha^2}\beta t^2 + \overline{\alpha^2}\beta^2 t^3}$$

We know now that with this choice of h_o and with all other h_k taken as zero, g_2 will be zero and the filter will contain no differentiators.

The calculated value of h_o when inserted into the expressions for d_1 and d_2 gives

$$d_1(t) = \frac{\beta\overline{\alpha^2}t(6B + \beta\overline{\alpha^2}t^3)}{6B\beta + 3\overline{\alpha^2}t + 3\overline{\alpha^2}\beta t^2 + \overline{\alpha^2}\beta^2 t^3}$$

$$d_2(t) = -\frac{3B\overline{\alpha^2}te^{-\beta t}(1 + \beta t)}{6B\beta + 3\overline{\alpha^2}t + 3\overline{\alpha^2}\beta t^2 + \overline{\alpha^2}\beta^2 t^3}$$

Therefore,

$$\left.\begin{aligned} g_1(t) &= \frac{6B\beta\overline{\alpha^2}t}{6B\beta + 3\overline{\alpha^2}t + 3\overline{\alpha^2}\beta t^2 + \overline{\alpha^2}\beta^2 t^3} \\ g_2(t) &= 0 \end{aligned}\right\} \tag{29}$$

By equation (21), then, the optimum impulse response will be

$$g(t,\tau) = \frac{g_1(t)}{2B\beta}\left[\beta^2\tau u(t - \tau) + (1 + \beta t)\delta(t - \tau) - \delta(\tau)\right] \tag{30}$$

where $g_1(t)$ is given by equation (29).

The error. The minimum error can be found using equation (7a); in fact,

$$E^2_{\min}(t) = \overline{\alpha^2}t^2 - \frac{3(\overline{\alpha^2})^2t^2}{6B\beta+3\overline{\alpha^2}t+3\overline{\alpha^2}\beta t^2+\overline{\alpha^2}\beta^2t^3}\int_0^t \tau[1+\beta t)\delta(t-\tau)+\beta^2\tau-\delta(\tau)]d\tau$$

$$= \overline{\alpha^2}t^2\left\{1 - \frac{3\overline{\alpha^2}}{6B\beta+3\overline{\alpha^2}t+3\overline{\alpha^2}\beta t^2+\overline{\alpha^2}\beta^2t^3}\left[t(1+\beta t) + \frac{\beta^2t^3}{3}\right]\right\}$$

$$= \frac{6B\beta\,\overline{\alpha^2}\,t^2}{6B\beta+3\overline{\alpha^2}t+3\beta\overline{\alpha^2}t^2+\overline{\alpha^2}\beta^2t^3}$$

$$\rightarrow \frac{6B}{\beta t} \text{ as } t \rightarrow \infty$$

The system equations.- Let us partition the impulse response (30) into two parts; we do this because the term $\delta(\tau)$ which occurs is essentially different from the others. Thus, we write

$$g(t,\tau) = k(t,\tau) - l(t,\tau)$$

where

$$k(t,\tau) = g_1(t)[\beta^2\tau\mu(t-\tau)+(1+\beta t)\delta(t-\tau)]$$

$$l(t,\tau) = g_1(t)\delta(\tau) \tag{31}$$

Each of these terms is an impulse response in its own right. We begin this section by finding the differential equation satisfied by the response k. Thus, we once again seek two functions $F(t,p)$ and $f(t,p)$, polynomials in $p = d/dt$, such that

$$F(t,p)k(t,\tau) = f(t,p)\delta(t-\tau)$$

The order of $F(t,p)$ is always equal to the number of terms in the second sum of equation (21); thus, in general, this order would be P. However, for this example, $g_2(t) = 0$, and so there is only one term left in this sum. Therefore, we set

$$F(t,p) = p+\xi_o(t)$$

There is exactly one δ function in the expression (31) for $k(t,\tau)$; it follows that $f(t,p)$ has the same order as F - that is, we may set

$$f(t,p) = \eta_1(t)p + \eta_0(t)$$

The differential equation we are seeking therefore reduces to

$$\frac{\partial}{\partial t} k(t,\tau) + \xi_0(t)k(t,\tau) = \eta_0(t)\delta(t-\tau) + \eta_1(t)\dot{\delta}(t-\tau) \tag{32}$$

As before, $\delta(t-\tau)$ is zero for $t > \tau$; therefore, we find

$$\xi_0(t) = -\frac{(\partial/\partial t)k(t,\tau)}{k(t,\tau)}$$

$$= -\frac{(\partial/\partial t)\beta^2\tau g_1(t)}{\beta^2\tau g_1(t)}$$

since $u(t-\tau) = 1$ for $t > \tau$. Thus,

$$\xi_0(t) = -\frac{\dot{g}_1(t)}{g_1(t)} \tag{33}$$

To find η_0 and η_1, substitute equation (31) into (32) with ξ_0 given by (33). This gives

$$g_1(t)[\beta(1+\beta\tau)\delta(t-\tau)+(1+\beta t)\dot{\delta}(t-\tau)] = \eta_0\delta(t-\tau) + \eta_1(t)\dot{\delta}(t-\tau)$$

Hence, just as in example I,

$$\eta_0(t) = \beta(1+\beta t)g_1(t)$$

$$\eta_1(t) = (1+\beta t)g_1(t)$$

Thus, the input-output relationship for that part of the optimum system described by the impulse response $k(t,\tau)$ can be written

$$\dot{x} - \frac{\dot{g}_1(t)}{g_1(t)}\, x = (1 + \beta t) g_1(t)\left(\beta i + \frac{di}{dt}\right) \qquad (34)$$

where $g_1(t)$ is given by equation (29).

It remains to discuss the impulse response $l(t,\tau)$. This response corresponds to a "memory" element, the output of which is put through a time-variable "gain." This is clear, for according to equation (2), the response of the system with impulse response l to any input i is

$$\int_0^t l(t,\tau) i(\tau) d\tau = \int_0^t g_1(t) \delta(\tau) i(\tau) d\tau$$

$$= g_1(t) i(0)$$

Thus, we find the optimum system to be that system whose response is the difference of the responses of the system described by equation (33) and the "memory-gain" system having the impulse response l.

The limiting case.- It was mentioned earlier that the noise described by the autocorrelation (24) is approximated by white noise when B and β are large. To see this, consider the integral equation (25). It is not hard to show that as $\beta \rightarrow \infty$,

$$\int_0^t e^{-\beta(\tau-\sigma)} g(t,\sigma) d\sigma = \frac{2}{\beta}\, g(t,\tau) + O\left(\frac{1}{\beta^2}\right)$$

Hence, if $B = N\beta/2$, the last two terms of the integral equation (25) approach $Ng(t,\tau)$ as $\beta \rightarrow \infty$, so that the integral equation itself resembles equation (12) more and more closely. This fact can be used as an additional check on our results, for the impulse response (30) should approach (13) as $\beta \rightarrow \infty$. This is the case, for with $B = N\beta/2$, then as $\beta \rightarrow \infty$,

$$\frac{g_1(t)}{2B\beta} \rightarrow \frac{3\overline{\alpha^2} t}{3N + \overline{\alpha^2} t^3}\, \frac{1}{\beta^2}$$

Hence, all the terms in equation (30) except the first approach zero, so that

$$g(t,\tau) \rightarrow \frac{3\overline{\alpha^2} t\tau}{3N + \overline{\alpha^2} t^3}\, u(t - \tau)$$

which agrees with equation (13).

Example IV

For our fourth example, we consider a problem in approximate differentiation. This will serve to illustrate the solution when the more general formulation leading to equation (6b) is used. The problem will be to find the slope of a measured line passing through the origin.

Calculation of the correlation functions.- Let the equation of the line be

$$m = \alpha t$$

where it is assumed that the slopes α have a certain probability distribution. We shall write $\overline{\alpha^2}$ for the mean square value of α. Since it is desired to find the slope of the line, the output of the filter should as closely as possible approximate the quantity

$$\mu = \frac{dm}{dt} = \alpha$$

About the noise, it will be assumed it is independent of the message and described by the autocorrelation function (24). Then

$$\varphi_{\mu\mu}(t,\tau) = \mathrm{Av}\left\{\alpha \cdot \alpha\right\}$$

$$= \overline{\alpha^2}$$

$$\varphi_{\mu i}(t,\tau) = \mathrm{Av}\left\{\alpha \cdot \alpha\tau\right\}$$

$$= \overline{\alpha^2}\tau$$

$$\varphi_{ii}(t,\tau) = \mathrm{Av}\left\{\alpha t \cdot \alpha\tau\right\} + Be^{-\beta(t-\tau)}$$

$$= \overline{\alpha^2}t\tau + Be^{-\beta|t-\tau|}$$

for $t,\tau \geq 0$.

The integral equation and its solution.- Substituting the above correlation functions into equation (6b), we see that the optimum impulse response must satisfy

$$\overline{\alpha^2}\tau = \overline{\alpha^2}\tau\int_0^t \sigma g(t,\sigma)d\sigma + Be^{-\beta\tau}\int_0^\tau e^{\beta\sigma}g(t,\sigma)d\sigma +$$

$$Be^{\beta\tau}\int_\tau^t e^{-\beta\sigma}g(t,\sigma)d\sigma \ , \qquad 0 \leq \tau \leq t \tag{35}$$

This equation is very similar to the earlier equation (25) and, in fact, it can be solved immediately by using the solution (30) of (25). Multiply equation (35) through by t; the resulting equation can then be written

$$\overline{\alpha^2}t\tau = \overline{\alpha^2}\tau\int_0^t \sigma[tg(t,\sigma)]d\sigma + Be^{-\beta\tau}\int_0^\tau e^{\beta\sigma}[tg(t,\sigma)]d\sigma +$$

$$Be^{\beta\tau}\int_\tau^t e^{-\beta\sigma}[tg(t,\sigma)]d\sigma \ , \qquad 0 \leq \tau \leq t$$

which is exactly the same as equation (25) with $g(t,\sigma)$ replaced by $tg(t,\sigma)$ throughout. Consequently, from equation (30), the impulse response satisfying (35) is

$$g(t,\tau) = \frac{g_1(t)}{2B\beta t}\left[\beta^2\tau\mu(t-\tau)+(1+\beta t)\delta(t-\tau) - \delta(\tau)\right] \tag{36}$$

where $g_1(t)$ is given by equation (29).

The error.- According to equation (7b), the mean square error of the optimum system will be

$$E^2_{min}(t) = \overline{\alpha^2} - \int_0^t \overline{\alpha^2}\tau g(t,\tau)d\tau$$

$$= \frac{6B\beta\overline{\alpha^2}}{6B\beta + 3\overline{\alpha^2}t + 3\overline{\alpha^2}\beta t^2 + \overline{\alpha^2}\beta^2 t^3}$$

$$\rightarrow \frac{6B}{\beta t^3} \text{ as } t \rightarrow \infty$$

Note that this implies that the error can be made as small as desired by measuring the line for large enough t.

The system equations.- As in example III, the optimum response is now divided in two parts:

$$g(t,\tau) = k(t,\tau) - l(t,\tau)$$

where

$$k(t,\tau) = \frac{g_1(t)}{2B\beta t}\left[\beta^2\tau\mu(t-\tau)+(1+\beta t)\delta(t-\tau)\right]$$

$$l(t,\tau) = \frac{g_1(t)}{2B\beta t}\,\delta(\tau)$$

The input-output relation corresponding to the impulse response k can be found by using the techniques previously described or by using equation (34). There results

$$t\dot{x} + \left(1 - \frac{\dot{g}_1}{g_1}\right)x = (1+\beta t)g_1(t)\left(\beta i + \frac{di}{dt}\right)$$

As before, the impulse response l corresponds to a memory-gain element.

The limiting case.- The above solution (36) simplifies if B and β are large. In this case, set $B = N\beta/2$ and let $\beta \rightarrow \infty$. Then,

$$g(t,\tau) \rightarrow \frac{3\overline{\alpha^2}\tau}{3N + \overline{\alpha^2}t^3}\,u(t-\tau)$$

THE CASE $w \equiv 0$

It is clear that one special case exists for which the method of the last section fails. This is the case when the function w of assumption (ii) is identically zero. The failure arises in the very first step, since the functions $\Gamma_p(s)$ cannot even be defined. A technique applicable when $w \equiv 0$ will now be described.

In this case, unfortunately, there is no method which can be followed mechanically in all cases. The reason for this is that if $w \equiv 0$, there may not be any solution. In fact, it can be shown that $w \equiv 0$ implies that there is either no solution at all or else there are infinitely many.

If no solution exists, it means, of course, that the original question was improperly posed and it was not an optimization problem at all. Hence, we shall assume that solutions exist.

To show how to find a solution, let

$$\left.\begin{aligned} \varphi_{ii}(t,\tau) &= \sum_{1}^{P} a_p(t)b_p(\tau) , \qquad \tau \leq t \\ \varphi_{\mu i}(t,\tau) &= \sum_{1}^{P} c_p(t)b_p(\tau) , \qquad \tau \leq t \end{aligned}\right\} \quad . \tag{37}$$

as before. We now assume that P functions $\gamma_p(t)$ can be found such that the determinant which has

$$I_{pq}(t) = \int_0^t a_p(\sigma)\gamma_q(\sigma)d\sigma$$

in its pth row and qth column is different from zero. Unfortunately, there are no set rules for determining these functions γ_n. If these are known, however, the optimum can be determined as follows.

Solve the following equations for the functions $g_n(t)$:

$$\sum_{q=1}^{P} I_{pq}(t)g_q(t) = d_p(t) , \qquad p = 1, \ldots, P \tag{38}$$

Then, the function

$$g(t,\tau) = \sum h_k(t)\delta^{(k)}(t - \tau) + \mu(t - \tau)\sum_{p=1}^{P} g_p(t)\gamma_p(\tau) \tag{21}$$

satisfies the integral equation provided the functions $d_p(t)$ are given by the equation

$$d_p(t) = c_p(t) - \sum_{k} h_k(t) \frac{d^k}{dt^k} a_p(t) , \qquad p = 1, \ldots, P \tag{19}$$

Example V

To illustrate the method just described, consider the somewhat artificial problem of measuring the lengths of a collection of rods. Suppose it is definitely known that these lengths are not less than A_1 nor more than A_2 inches, and that our measuring device is a ruler which can be read with an accuracy of $\pm B$ inches; in this case, the messages, that is, the lengths of the rods, can be interpreted as steps applied at time zero, of magnitude α, where $A_1 \leq \alpha \leq A_2$. Similarly, the noise, that is, the least reading of the ruler, can be interpreted as a step of magnitude β where $-B \leq \beta \leq B$. When viewed in this light, the problem is amenable to the techniques of this report.

The correlation functions.- Assume that all values of α and β in their respective ranges are equally likely. Then, for $t,\tau \geq 0$,

$$
\begin{aligned}
\varphi_{mm}(t,\tau) &= \mathrm{Av}\left\{m(t;\alpha)m(\tau;\alpha)\right\} \\
&= \mathrm{Av}\left\{\alpha \cdot \alpha\right\} \\
&= \frac{1}{A_2 - A_1}\int_{A_1}^{A_2} \alpha^2 d\alpha \\
&= \frac{A_1^2 + A_1 A_2 + A_2^2}{3}
\end{aligned}
$$

Setting $A^2 = A_1^2 + A_1 A_2 + A_2^2$, we have

$$\varphi_{mm}(t,\tau) = \frac{A^2}{3}$$

Assuming α and β independent, we see that

$$
\begin{aligned}
\varphi_{mi}(t,\tau) &= \mathrm{Av}\left\{\alpha(\alpha + \beta)\right\} \\
&= \mathrm{Av}\left\{\alpha^2\right\} \\
&= \frac{A^2}{3}
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
\varphi_{ii}(t,\tau) &= \mathrm{Av}\left\{(\alpha + \beta)(\alpha + \beta)\right\} \\
&= \frac{A^2}{3} + \frac{B^2}{3}
\end{aligned}
\tag{40}
$$

<u>The integral equation and its solution</u>.- Substituting the functions φ_{mi} and φ_{ii} into equation (6a) gives for the integral equation for the optimum

$$\frac{A^2}{3} = \frac{A^2 + B^2}{3} \int_0^t g(t,\sigma)\,d\sigma \tag{41}$$

which clearly is satisfied by

$$g(t,\tau) = \frac{A^2}{A^2 + B^2}\,\delta(t - \tau) \tag{42}$$

Thus, a solution of equation (41) can be found by inspection; we now attempt to find a solution by the method described on page 34. In order to illustrate a difficulty which sometimes arises when applying any of the methods which have been described, we shall begin with a wrong procedure.

Comparing equations (37) with equations (39) and (40), we conclude that we may write $P = 2$, and

$$a_1(t) = \frac{A^2}{3}, \qquad b_1(t) = 1, \qquad c_1(t) = \frac{A^2}{3}$$

$$a_2(t) = \frac{B^2}{3}, \qquad b_2(t) = 1, \qquad c_2(t) = 0$$

However, with this choice of these functions, no solution appears to exist, for no matter what γ_1 and γ_2 may be

$$\begin{vmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{vmatrix} = \begin{vmatrix} \frac{A^2}{3}\int_0^t \gamma_1(\tau)\,d\tau & \frac{A^2}{3}\int_0^t \gamma_2(\tau)\,d\tau \\ \frac{B^2}{3}\int_0^t \gamma_1(\tau)\,d\tau & \frac{B^2}{3}\int_0^t \gamma_2(\tau)\,d\tau \end{vmatrix} = 0$$

Thus, we might conclude from what has gone before that no solution exists; however, this apparent nonexistence of a solution does not withstand close analysis, since we know that there is at least one solution - given by equation (42). The difficulty encountered is due to the fact that the functions $a_1(t)$ and $a_2(t)$ as chosen above are linearly dependent. In order to arrive at a correct solution, it is always essential that the functions $a_1(t)$ and $a_2(t)$ as well as the functions $b_1(t)$ and $b_2(t)$ be chosen so that they are independent.

Perhaps the simplest correct solution of the problem results by taking $P = 1$ and setting

$$a_1(t) = \frac{A^2 + B^2}{3}, \qquad b_1(t) = 1, \qquad c_1(t) = \frac{A^2}{3}$$

Then, there is only one function I_{pq}, notably

$$I_{11}(t) = \frac{A^2 + B^2}{3} \int_0^t \gamma_1(\sigma)d\sigma$$

From equation (38), we then conclude that whatever γ_1 may be,

$$g_1(t) = \frac{d_1(t)}{\frac{A^2 + B^2}{3} \int_0^t \gamma_1(\sigma)d\sigma}$$

$$= \frac{A^2 - (A^2 + B^2)h_o(t)}{(A^2 + B^2) \int_0^t \gamma_1(\sigma)d\sigma}$$

by equation (20). For the usual reasons, we now set $h_k(t) \equiv 0$, $k \geq 1$, to obtain from equation (21)

$$g(t,\tau) = h_o(t)\delta(t - \tau) + u(t - \tau) \frac{A^2 - (A^2 + B^2)h_o(t)}{(A^2 + B^2) \int_0^t \gamma_1(\sigma)d\sigma} \gamma_1(\tau) \tag{43}$$

This function optimizes the system whatever $h_o(t)$ may be; clearly the choice of $h_o(t)$ which most simplifies $g(t,\tau)$ is

$$h_o(t) = \frac{A^2}{A^2 + B^2}$$

which results in

$$g(t,\tau) = \frac{A^2}{A^2 + B^2} \delta(t - \tau)$$

in accordance with equation (42).

The error.- According to equation (7a), the mean square error of the optimum system is

$$E^2_{min}(t) = \frac{A^2}{3} - \int_0^t \left[\frac{A^2}{A^2+B^2}\,\delta(t-\tau)\right]\frac{A^2}{3}\,d\tau$$

$$= \frac{B^2}{3}\,\frac{1}{1+(B/A)^2}$$

$$\approx \frac{B^2}{3}, \qquad \text{if } \frac{B}{A} \ll 1$$

It should be noted, incidentally, that it can be shown that all systems described by equation (43), no matter what $h_o(t)$ may be, have the same error; of course, this was implied previously by calling all functions (43) "optima." To see this, consider the error corresponding to the impulse response (43). We have

$$E^2(t) = \frac{A^2}{3} - \frac{A^2}{3}\int_0^t \left[h_o(t)\delta(t-\tau) + u(t-\tau)\,\frac{A^2-(A^2+B^2)h_o(t)}{(A^2+B^2)\int_0^t \gamma_1(\sigma)d\sigma}\,\gamma_1(\tau)\right]d\tau$$

$$= \frac{A^2}{3} - \frac{A^2}{3}\left[h_o(t) + \frac{A^2-(A^2+B^2)h_o(t)}{(A^2+B^2)\int_0^t \gamma_1(\sigma)d\sigma}\int_0^t \gamma_1(\tau)d\tau\right]$$

$$= \frac{A^2}{3} - \frac{A^2}{3}\left[h_o(t) + \frac{A^2}{A^2+B^2} - h_o(t)\right]$$

$$= \frac{B^2}{3}\,\frac{1}{1+(B/A)^2}$$

$$= E^2_{min}(t)$$

The system equation.- It is obvious that the system with impulse response (42) is a simple gain. Thus, the output is obtained from the input by multiplication of the latter by the gain $A^2/(A^2+B^2)$. To find the best estimate (according to the present criterion) of the lengths of the rods, the measured lengths should be multiplied by $A^2/(A^2+B^2)$.

This result, that the optimum system is simply represented by a gain, is quite general, being true whenever the ratio $\varphi_{\mu i}/\varphi_{ii}$ is a constant. In such a case, the optimum system is a gain adjusted so that the mean square of the output is the same as that of the desired quantity μ.

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